4

Wave Polarization

4-1 Introduction Consider a plane wave traveling out of the page (in the positive z direction), as in Fig. 4-1, with electric-field components in the x and y directions as given by†

$$E_x = E_1 \sin(\omega t - \beta z) \tag{4-1}$$
$$E_y = E_2 \sin(\omega t - \beta z + \delta) \tag{4-2}$$

where E_1, E_2 = constants

$\omega = 2\pi\nu$

$\beta = 2\pi/\lambda$

δ = phase difference of E_y and E_x

Equations (4-1) and (4-2) describe two *linearly polarized waves*, one polarized in the x direction and the other in the y direction.

Combining (4-1) and (4-2) vectorially, we obtain for the total or resultant field

$$\mathbf{E} = \mathbf{x}E_x + \mathbf{y}E_y \tag{4-3}$$

where $\mathbf{x}, \mathbf{y}$ = unit vectors in x and y directions

It follows that

$$\mathbf{E} = \mathbf{x}E_1 \sin(\omega t - \beta z) + \mathbf{y}E_2 \sin(\omega t - \beta z + \delta) \tag{4-4}$$

At $z = 0$, $E_x = E_1 \sin \omega t$ and $E_y = E_2 \sin(\omega t + \delta)$. Expanding E_y yields

$$E_y = E_2(\sin \omega t \cos \delta + \cos \omega t \sin \delta) \tag{4-5}$$

From the relation for E_x we have

$$\sin \omega t = \frac{E_x}{E_1} \tag{4-6}$$

and

$$\cos \omega t = \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2} \tag{4-7}$$

Introducing (4-6) and (4-7) in (4-5), time is eliminated, and we obtain, on rearranging, that

† For a more general discussion see Kraus (1950).

$$\frac{E_x^2}{E_1^2} - \frac{2E_xE_y \cos \delta}{E_1E_2} + \frac{E_y^2}{E_2^2} = \sin^2 \delta \tag{4-8}$$

or

$$aE_x^2 - bE_xE_y + cE_y^2 = 1 \tag{4-9}$$

where $a = 1/E_1^2 \sin^2 \delta$

$b = 2 \cos \delta / E_1E_2 \sin^2 \delta$

$c = 1/E_2^2 \sin^2 \delta$

Equation (4-9) may be recognized as the equation for an ellipse in its most general form, the axes of the ellipse, in general, not coinciding with

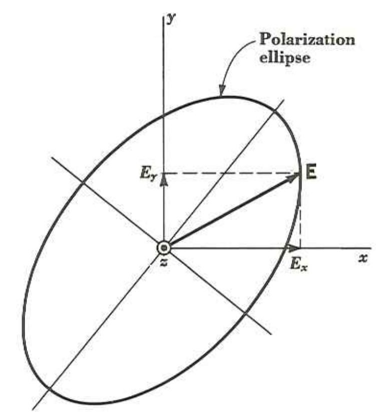

Fig. 4-1. Relation of instantaneous electric-field vector **E** to polarization ellipse.

the x or y axes. Thus, (4-4) represents the general case of elliptical polarization, the locus of the tip of the electric-field vector **E** describing an ellipse, as in Fig. 4-1.

Referring to Fig. 4-2, the line segment OA is the semimajor axis, and the line segment OB is the semiminor axis of the ellipse. The tilt angle of the ellipse is τ. The ratio of OA to OB is called the *axial ratio* (AR) *of the polarization ellipse* or simply the *axial ratio*. Thus,

$$\text{AR} = \frac{OA}{OB} \qquad 1 \leq \text{AR} \leq \infty \tag{4-10}$$

If $E_1 = 0$, the wave is *linearly polarized* in the y direction. If $E_2 = 0$, the wave is linearly polarized in the x direction. If $\delta = 0$ and $E_1 = E_2$, the wave is also linearly polarized but in a plane at angle of 45° with respect to the x axis. A further special case of interest occurs when $E_1 = E_2$ and $\delta = \pm 90°$. The resulting wave is *circularly polarized*. When $\delta = +90°$, the wave is said to be *left circularly polarized* and, when $\delta = -90°$, it is said to be *right circularly polarized*. Thus, from (4-4) we have for

$\delta = +90°$, at $z = 0$ and $t = 0$, that $E_x = 0$ and $\mathbf{E} = \mathbf{y}E_2$, as in Fig. 4-3*a*. Under the same conditions but at a later time such that $\omega t = 90°$, $E_y = 0$ and $\mathbf{E} = \mathbf{x}E_1$, as in Fig. 4-3*b*. The rotation of the electric-field vector is thus clockwise with the wave approaching. According to the IRE standards (1942) this sense of rotation is defined as left circular polarization. According to the older usage of classical physics this sense of rotation (clockwise with wave approaching) is defined as right circular polarization, or opposite to the IRE definition.

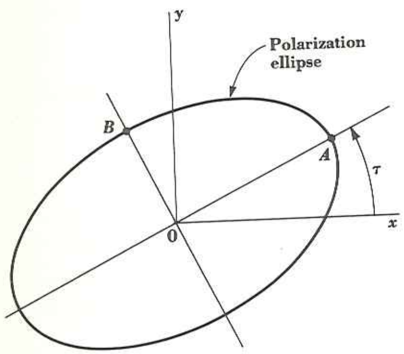

Fig. 4-2. Polarization-ellipse geometry

If the wave is viewed receding (from negative z axis in Fig. 4-1), the electric vector appears to rotate in the opposite direction. Hence, clockwise rotation with the wave receding is the same as counterclockwise rotation with the wave approaching. In the following the IRE definition will be used, since it could also be defined (without reference to the wave direction) by means of helical-beam antennas (Kraus, 1950). Thus, a right-handed helical-beam antenna radiates or receives right circular (IRE) polarization. A right-handed helix, like a right-handed screw, is right-handed regardless of the position from which the helix is viewed. There is no possibility here of ambiguity. The different definitions for the two types of circular polarization are summarized in Table 4-1.

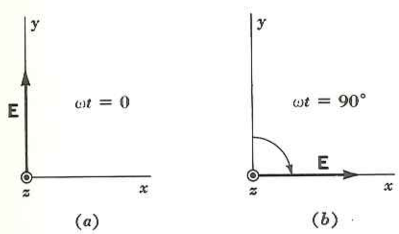

Fig. 4-3. Change in direction of **E** for left circular polarization. Time $t = 0$ in (*a*) and $\omega t = 90°$ in (*b*).

Table 4-1†

Polarization	*Classical physics usage*	*IRE definition (1942)*	*Type of helical-beam antenna for generating or receiving*
Clockwise (wave approaching) or counterclockwise (wave receding)	Right	Left	Left-handed
Counterclockwise (wave approaching) or clockwise (wave receding)	Left	Right	Right-handed

† A left circularly polarized wave (IRE) has an instantaneous electric field (as a function of space) that describes a right-handed screw.

In (4-4) the general case of an elliptically polarized wave was described in terms of two linearly polarized components. It is also possible to describe the general situation in terms of two circularly polarized waves of unequal amplitude. Thus, at $z = 0$, let

$$E_r = E_R e^{j\omega t} \tag{4-11}$$

and

$$E_l = E_L e^{-j(\omega t + \delta')} \tag{4-12}$$

where E_r = right circularly polarized wave (Fig. 4-4)
E_l = left circularly polarized wave
E_R, E_L = constants
δ' = phase difference

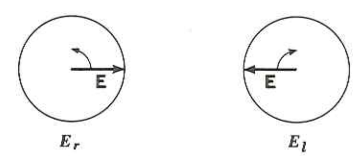

Fig. 4-4. Right and left circularly polarized waves.

Then the instantaneous linearly polarized components of the wave (E_x and E_y) are given by

$$E_x = \text{Re}\,(E_r + E_l) \tag{4-13}$$

and

$$E_y = \text{Im}\,(E_r + E_l) \tag{4-14}$$

or

$$E_x = E_R \cos \omega t + E_L \cos (\omega t + \delta') \tag{4-15}$$

and

$$E_y = E_R \sin \omega t - E_L \sin (\omega t + \delta') \tag{4-16}$$

On eliminating ωt, as done in deriving (4-8), (4-15) and (4-16) may be reduced to an equation having the form of an ellipse, demonstrating that (4-11) and (4-12) represent an elliptically polarized wave.

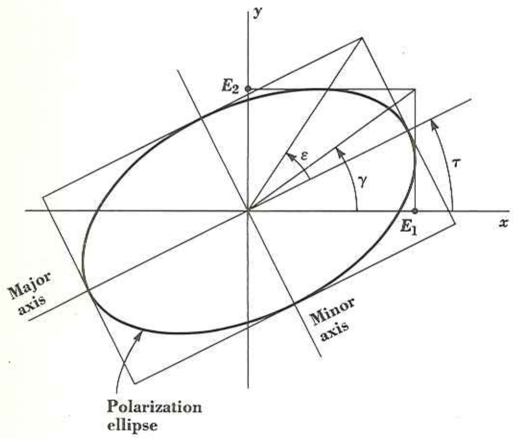

Fig. 4-5. Relation of amplitudes E_1 and E_2 and angles ϵ, γ, and τ to the polarization ellipse.

4-2 The Polarization Ellipse and the Poincaré Sphere The general case of an elliptically polarized wave may be described as before (at $z = 0$) by

$$E_x = E_1 \sin \omega t \tag{4-17}$$

and

$$E_y = E_2 \sin (\omega t + \delta) \tag{4-18}$$

where δ = phase difference between E_y and E_x $(-180° \leq \delta \leq +180°)$
Referring to Fig. 4-5, let

$$\gamma = \tan^{-1} \frac{E_2}{E_1} \qquad 0° \leq \gamma \leq 90° \tag{4-19}$$

where E_2/E_1 = amplitude ratio
Also let the *tilt angle* of the polarization ellipse be designated by τ, where $0° \leq \tau \leq 180°$, and let

$$\epsilon = \cot^{-1} (\mp \text{AR}) \qquad -45° \leq \epsilon \leq +45° \tag{4-20}$$

where $\text{AR} = \dfrac{\text{major axis}}{\text{minor axis}} \qquad 1 \leq |\text{AR}| \leq \infty$

with the minus sign used for right-handed and the plus sign for left-handed (IRE) polarization.

The above quantities are interrelated by the following equations (Poincaré, 1892; Deschamps, 1951)

$$\cos 2\gamma = \cos 2\epsilon \cos 2\tau \tag{4-21}$$

$$\tan \delta = \frac{\tan 2\epsilon}{\sin 2\tau} \tag{4-22}$$

or

$$\tan 2\tau = \tan 2\gamma \cos \delta \tag{4-23}$$

$$\sin 2\epsilon = \sin 2\gamma \sin \delta \tag{4-24}$$

Knowing ϵ and τ, one can determine γ and δ using (4-21) and (4-22). Conversely, knowing γ and δ, one can find ϵ and τ by means of (4-23) and (4-24). It is convenient to describe the *polarization state* by either of the two sets of angles ϵ and τ or γ and δ. Let the polarization state as a function of ϵ and τ be designated by $M(\epsilon,\tau)$ or simply M and the polarization state as a function of γ and δ be designated by $P(\gamma,\delta)$ or simply P. Then on the Poincaré sphere, one-eighth of which is shown in Fig. 4-6, the angles ϵ, τ, γ,

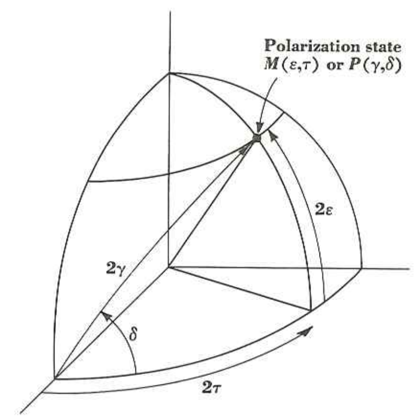

Fig. 4-6. Poincaré sphere.

and δ are related as indicated. More specifically, any point on the sphere describes a particular polarization state. In terms of $M(\epsilon,\tau)$ its coordinates are

$$2\epsilon = \text{latitude} \qquad -90° \leq 2\epsilon \leq +90°$$

and

$$2\tau = \text{longitude} \qquad 0° \leq 2\tau \leq 360°$$

while in terms of $P(\gamma,\delta)$ its coordinates are

$$2\gamma = \text{great-circle distance from origin} \qquad 0° \leq 2\gamma \leq 180°$$

and

$$\delta = \text{angle of great-circle line with respect to equator} \quad -180° \leq \delta \leq +180°$$

Several special cases are of interest.

Case 1. For $\delta = 0$ or $\pm 180°$, E_x and E_y are exactly in phase or out of phase. Thus, any point on the equator represents a state of linear polarization. At the origin the polarization is linear and horizontal ($\tau = 0$) as in Fig. 4-7. On the equator at 90° to the right the polarization is linear with a tilt angle of 45°, while at 180° from the origin the polarization is linear and vertical ($\tau = 90°$), etc.†

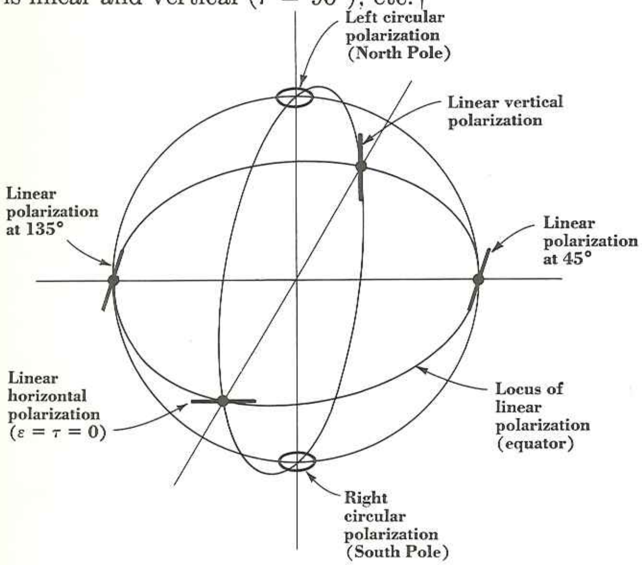

Fig. 4-7. Polarization at cardinal points of Poincaré sphere.

Case 2. For $\delta = \pm 90°$ and $E_2 = E_1$ $(2\gamma = 90°)$, E_x and E_y have equal amplitudes but are in phase quadrature, which is the condition for circular polarization. Thus, the poles represent a state of circular polarization, the north pole representing left circular polarization and the south pole right circular polarization (IRE), as suggested in Fig. 4-7.

Cases 1 and 2 represent limiting conditions. In the general case any point in the northern hemisphere describes a left elliptically polarized wave ranging from pure left circular at the pole to linear at the equator. Likewise, any point in the southern hemisphere describes a right elliptically polarized wave ranging from pure right circular at the pole to linear at the equator.

† Strictly speaking, the term *horizontal* polarization for the $\tau = 0°$ case and *vertical* polarization for the $\tau = 90°$ case is significant only for the special case where the wave is propagating parallel to the ground so that the x- axis is horizontal and the y- axis is vertical.

4-3 The Response of an Antenna to a Wave of Arbitrary Polarization

The response of a receiving antenna, given by its terminal voltage V, when a wave of field intensity **E** is incident upon it may be expressed as

$$V = \mathbf{E} \cdot \mathbf{l} = El \cos \theta \qquad (4\text{-}25)$$

where $\mathbf{l}$ = *effective length* of antenna
θ = angle between **E** and **l**

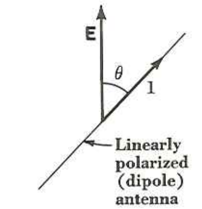

Fig. 4-8. Dipole of effective length **l** and incident wave of field intensity **E**.

In early antenna work l was commonly referred to as the *effective height.* The relation between **E** and **l** for a linearly polarized wave and a linearly polarized antenna (dipole) is suggested in Fig. 4-8. For the general case of an elliptically polarized wave the magnitude of **E** is given by

$$E = \sqrt{E_1^2 + E_2^2} \qquad (4\text{-}26)$$

and the response V of a linearly polarized (dipole) antenna to such a wave may be as suggested in Fig. 4-9.

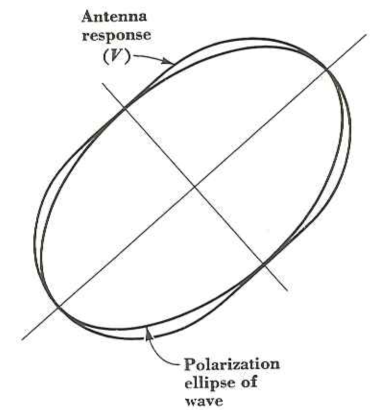

Fig. 4-9. Response of linearly polarized antenna to an elliptically polarized wave.

Let the polarization state of the antenna be designated as $M_a(\epsilon,\tau)$ or simply M_a, and the polarization state of the wave as $M(\epsilon,\tau)$ or simply M. The polarization state of the antenna is defined as the polarization state of the wave radiated by the antenna when it is transmitting. Then we have that

$$V = El \cos \frac{MM_a}{2} \qquad (4\text{-}27)$$

where MM_a is the great-circle distance between points (or polarization states) M and M_a on the Poincaré sphere.

Several special cases are of interest.

Case 1. If $MM_a = 0$, the antenna is matched to the wave, and $V = El$.

Case 2. If the wave is left circularly polarized and the antenna is right circularly polarized, $MM_a = 180°$ and $V = 0$.

Case 3. If the wave is vertically polarized and the antenna is horizontally polarized, $MM_a = 180°$ and $V = 0$.

Cases 2 and 3 are illustrations of the fact that an antenna is blind to a wave of the antipodal polarization state.

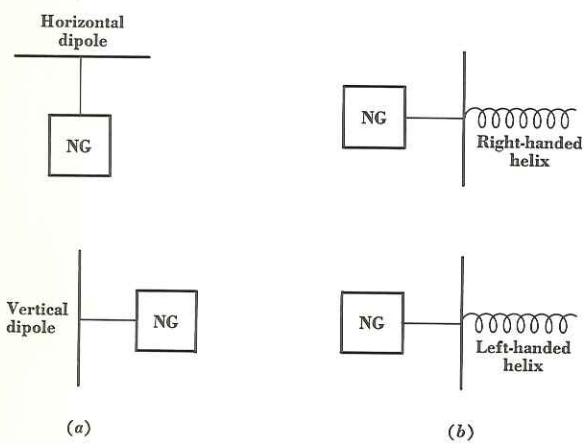

Fig. 4-10. Methods of generating a completely unpolarized wave. At (*a*) two independent noise generators are connected to vertical and horizontal dipoles (wave out of page). At (*b*) two independent noise generators are connected to helical-beam antennas of opposite hand (wave to right).

4-4 Partial Polarization and the Stokes Parameters The foregoing sections deal with completely polarized waves, where E_1, E_2, and δ are constants (or at least slowly varying functions of time). The radiation from a monochromatic (single-frequency) transmitter is of this type. However, in general, the emission from celestial radio sources extends over a wide frequency range and within any finite bandwidth $\Delta\nu$ consists of the superposition of a large number of statistically independent waves of a variety of polarizations. The resultant wave is said to be *randomly polarized.* For such a wave we may write

$$E_x = E_1(t) \sin \omega t \tag{4-28}$$
$$E_y = E_2(t) \sin [\omega t + \delta(t)] \tag{4-29}$$

where all the time functions are independent. The time variations of $E_1(t)$, $E_2(t)$, and $\delta(t)$ are slow compared to that of the mean frequency, ν ($\omega = 2\pi\nu$) being of the order of the bandwidth $\Delta\nu$.

A wave of this type could be generated by connecting one noise generator to a horizontally polarized antenna (dipole) and a second noise generator to a vertically polarized antenna (dipole), as in Fig. 4-10*a*. An alternative scheme would be to use two noise generators and a left- and right-handed helical-beam antenna producing left and right circular polarization, as in Fig. 4-10*b*.

The most general situation is one in which the wave is *partially polarized*; i.e., it may be considered to be of two parts, one completely polarized and the other completely unpolarized. A *completely unpolarized* (or completely randomly polarized) wave results if the powers radiated from the two generators in Fig. 4-10*a* or *b* are equal. The waves emitted by celestial radio sources are generally of the partially polarized type, tending in many cases to completely unpolarized radiation but in other cases to a significant amount of polarization.

To deal with partial polarization it is convenient to use the Stokes parameters introduced by Sir George Stokes (1852) (see Chandrasekhar, 1950). As an introduction let us first consider their application to a completely polarized wave.

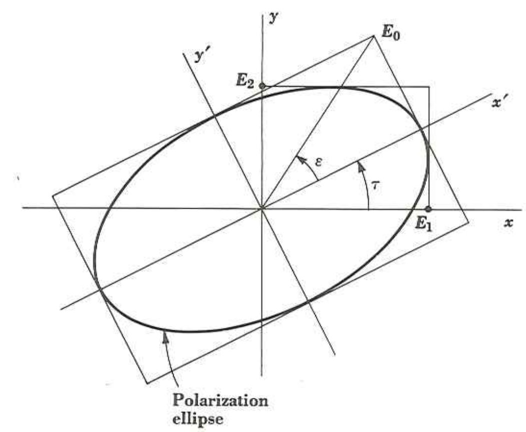

Fig. 4-11. Relation of polarization-ellipse axes (x',y') to reference axes (x,y).

Referring to Fig. 4-11, we can write

$$E_x = E_1 \sin(\omega t - \delta_1) \tag{4-30}$$
$$E_y = E_2 \sin(\omega t - \delta_2) \tag{4-31}$$

where $\delta_1 - \delta_2$ = phase difference of E_x and E_y