expect $\mu_L = \cos\theta_L \simeq (1-\theta_L^2/2)$, with $\theta_L \ll 1$. Comparing with the relation $1-(v/c) \simeq (2\gamma^2)^{-1}$ we find that $\theta_L \simeq \gamma^{-1} \ll 1$. Hence this direction of propagation is almost along the positive x axis in the lab frame. However, because $\mu_R = 0$, it is clear that $\theta_R = \pi/2$ and the wave is propagating perpendicularly to the x axis in the rest frame. Thus a wave emitted in the direction perpendicular to the direction of motion will be turned around to (almost) forward direction by the relativistic motion of the source. Similarly, when $\mu_L = 0$, $\mu_R = -(v/c)$. In this case, a wave that has been travelling almost along the negative x axis has been turned around to travel orthogonally to the x axis. It is clear from the diagram that the wave will appear to propagate forward in both frames only if $\theta_L < \gamma^{-1}$.

In the two cases discussed above, the frequency of the wave changes. When $\mu_L = (v/c)$, $\omega_R = \omega_L\gamma^{-1}$ and the wave is still blue shifted. When $\mu_L = 0$, $\omega_R = \gamma\omega_L$ and the wave is red shifted. To find the angle of propagation at which there is no frequency change, we have to set $\omega_L = \omega_R$ and solve for μ. This gives $\theta_c \simeq \gamma^{-1/2}$. For $\theta < \theta_c$ the wave is blue shifted. A wave propagating along $\theta = \theta_c$ appears to make the same angle with respect to the x axis in both the frames.

The discussion above shows that the motion of a source drags the wave forward. (A corollary to this result is that a charged particle, moving relativistically, will beam most of the radiation it emits in the forward direction.) Correspondingly, a charged particle moving relativistically through an isotropic bath of radiation will see most of the radiation as hitting it in the front.

Finally, by using Eq. (3.140), we can find the energy-momentum tensor for the plane wave that is given by

$$T^{ab} = \frac{Wc^2}{\omega^2}k^a k^b, \tag{3.170}$$

where $W = (E^2 + B^2)/8\pi = E^2/4\pi$. Because the left-hand side of Eq. (3.170) and $k^a k^b$ are tensors, it follows that the combination (E/ω) is Lorentz invariant. (This is an interesting result, especially as it is valid for any LT – not merely for the ones along the direction of propagation.) The momentum flux of a plane wave is given by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\mathbf{E}\times(\mathbf{n}\times\mathbf{E}) = \frac{c}{4\pi}E^2\hat{\mathbf{n}} = \frac{c}{4\pi}B^2\hat{\mathbf{n}}, \tag{3.171}$$

or $\mathbf{S} = cW\mathbf{n}$. The relation between momentum density and energy density $S/c^2 = W/c$ is the same as that for a particle of mass zero moving with the speed of light.

3.12.2 Polarisation of Light

The vector potential for the plane wave has the form given in Eq. (3.162). Since $a^0 = 0$ it follows that $\phi = 0$. (For convenience we omit the subscript $\mathbf{k}$ in the notation). The corresponding electric and magnetic fields are given by

$\mathbf{E} = ik\mathbf{A}$, $\mathbf{B} = i\mathbf{k}\times\mathbf{A}$. More explicitly, the electric field is

$$\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]. \tag{3.172}$$

Such a field has definite polarisation, which we now discuss.

Because $\mathbf{E}_0$ is a complex vector, so also is its square, which we write as $\mathbf{E}_0^2 = |\mathbf{E}_0|^2 \exp(-2i\alpha)$. Defining a complex vector $\mathbf{b}$ by $\mathbf{E}_0 \equiv \mathbf{b}e^{-i\alpha}$, we see that $\mathbf{b}^2 = |\mathbf{E}_0|^2$ is real; if $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2$, where $\mathbf{b}_{1,2}$ are real, it follows that $\mathbf{b}_1\cdot\mathbf{b}_2 = 0$ so that the two vectors are perpendicular. Further, because the electric field is orthogonal to the direction of propagation of the wave (which is, say, the x axis), we can take $\mathbf{b}_1$ along the y axis and $\mathbf{b}_2$ along the z axis. Finally, noting that the physical electric-field components are the real parts of the complex exponents, we can write the electric field as

$$\begin{aligned} E_y &= b_1 \cos(\omega t - \mathbf{k}\cdot\mathbf{x} + \alpha), \\ E_z &= \pm b_2 \sin(\omega t - \mathbf{k}\cdot\mathbf{x} + \alpha). \end{aligned} \tag{3.173}$$

This gives the relation

$$\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1 \tag{3.174}$$

between the components of the wave, showing that the tip of the vector rotates on an ellipse in the x–y plane as t varies. Such a wave is called an elliptically polarised wave. If $b_1 = b_2$, the ellipse becomes a circle, and we have a circularly polarised wave. If b_1 or b_2 vanishes, the field is along one of the axes and is called plane polarised.

The monochromatic wave is necessarily polarised in a manner discussed above. However, we often come across in nature sources of radiation that are not strictly monochromatic but contain frequencies in a narrow band $\Delta\omega$ around the mean frequency. In that case, the time variation of the electric field will be of the form $\mathbf{E} = \mathbf{E}_0(t)\exp(-i\omega t)$, where $\mathbf{E}_0(t)$ is a slowly varying function of time [compared to $\exp(-i\omega t)$]. To determine the degree of polarisation of such a wave, we should consider expressions that are quadratic in the electric field; however, it is now necessary to average these expressions over the rapidly varying part to determine the mean state of polarisation. The quadratic expressions are made of $E_\alpha E_\beta$, $E_\alpha E_\beta^*$, or their complex conjugates. Of these, the first one and its conjugate vary with a frequency 2ω and hence will average to zero. Therefore the polarisation properties are decided by the average of the product $E_\alpha E_\beta^* = E_{0\alpha}E_{0\beta}^*$.

We define a quantity $J_{\alpha\beta} = \langle E_{0\alpha}E_{0\beta}^*\rangle$ that has four independent components (because the indices take the values 1,2 in the plane perpendicular to the direction of propagation). The trace of this quantity, $J \equiv \sum J_{\alpha\alpha} = \langle\mathbf{E}_0\cdot\mathbf{E}_0\rangle$ measures the energy density of the field and is not directly related to the polarisation. Hence it is convenient to divide $J_{\alpha\beta}$ by its trace and define a polarisation tensor as

$\rho_{\alpha\beta} = (J_{\alpha\beta}/J)$. From this definition it follows that $\rho_{\alpha\beta} = \rho^*_{\beta\alpha}$, that is, the matrix is Hermitian with unit trace. Any such matrix can be written in the form

$$\rho_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} 1+\xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1-\xi_3 \end{bmatrix}, \tag{3.175}$$

where the quantities ξ_μ are called the Stokes parameters. From the definition, it is clear that the determinant of $\rho_{\alpha\beta}$, which is given by

$$\det \rho = \frac{1}{4}(1-\xi_1^2-\xi_2^2-\xi_3^2) \equiv \frac{1}{4}(1-P^2), \tag{3.176}$$

is positive (see Exercise 3.18). So each of the Stokes parameters vary in the range $(-1, 1)$.

Some general properties of the polarisation tensor can be easily ascertained. To begin with, if the wave is completely polarised, $\mathbf{E}_0$ is independent of t and the averaging has no effect on the definition of the polarisation tensor; in this case, the polarisation tensor is expressible as the direct product of two vectors. The necessary and sufficient condition for this is that $\det|\rho|$ vanishes, implying that $P=1$. On the other hand, a completely unpolarised wave will have – by symmetry – the polarisation tensor $\rho_{\alpha\beta} = (1/2)\delta_{\alpha\beta}$, so that $P=0$. Because of this feature P is called the degree of polarisation.

In general, the quantity ξ_2 represents the degree of circular polarisation; ξ_3 gives the degree of linear polarisation along the y or the z axis, with $\xi_3=1$ representing linear polarisation along the y axis and $\xi_3=-1$ giving linear polarisation along the z axis. The parameter ξ_1 quantifies the linear polarisation along directions that make 45° with the y axis; a value of $\xi_1=1$ corresponds to complete polarisation along $\phi=(\pi/4)$ and $\xi_1=-1$ corresponds to polarisation along $\phi=-(\pi/4)$.

Exercise 3.18
Properties of the polarisation tensor: Prove that the polarisation tensor must have positive determinant. Verify the various results stated above explicitly.

Exercise 3.19
Stokes parameters: Another way of characterising the polarisation of a wave is by using certain variables defined along the following lines. Consider an elliptically polarised light for which the major axis of the ellipse is inclined at an angle ψ with the arbitrarily chosen x axis. Let the amplitude of the electric field along the major and the minor axes of the ellipse be E_a and E_b with $\chi \equiv tan^{-1}(E_a/E_b)$. We now define the quantities S_i $(i=0,\ldots,3)$ and three variables (U, V, W) by the relations

$$\begin{aligned} S_0 = I = E_a^2 + E_b^2, \quad S_1 = Q = S_0\cos 2\chi\cos 2\psi, \\ S_2 = U = S_0\cos 2\chi\sin 2\psi, \quad S_3 = V = S_0 \sin 2\chi. \end{aligned} \tag{3.177}$$

(1) Show that the parameters S_i are not all independent. (2) Consider a sphere of radius S_0 in the space with Cartesian axes (S_1, S_2, S_3). Every point on the sphere (called a Poincaré sphere) defines a state of polarisation. Where do the following lie on the Poincare sphere: right circularly polarised light, left circularly polarised light, linearly polarised light? How does this description in terms of S_i relate to the one given in the text?

Exercise 3.20
Angular momentum of the wave: A circularly polarised electromagnetic wave of frequency ω impinges on a charged particle. Average the motion of the charged particle (which is assumed to be nonrelativistic) over a time T that is large compared with the period of the wave and show that the wave transfers an amount of energy $\mathcal{E}$ and angular momentum J to the charged particle, where $J=(\mathcal{E}/\omega)$.

3.13 Diffraction

The propagation of free electromagnetic waves is completely described by Eq. (3.164). By reformulating this equation in a more convenient form, we can understand a host of optical phenomena in which the wave nature of the light plays a vital role. We begin with the first of these phenomena, which is usually called diffraction.

Consider a monochromatic wave of frequency ω. Because the vector nature of the electromagnetic field is not very important in our discussion, we deal with just one component of the vector potential. In accordance with Eq. (3.164), any one component of the vector potential can be represented as

$$A(t,\mathbf{x}) = \int F_1(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t}\frac{d^3k}{(2\pi)^3}. \tag{3.178}$$

In the study of optical phenomena we are often concerned with waves that are propagating, by and large, in some given direction, which can be taken as the positive z axis. Mathematically, this means that the function $F_1(\mathbf{k})$ is significantly nonzero only for wave vectors with $k_z > 0$ and $(k_x, k_y) \ll k_z$. Further, because the wave has a definite frequency ω, the magnitude of the wave vector is fixed at the value ω/c. It follows that one of the components of the wave vector, say k_z, can be expressed in terms of the other three. Therefore the function F_1 has the structure

$$F_1(k_z, \mathbf{k}_\perp) = 2\pi f(\mathbf{k}_\perp)\delta_D\left(k_z - \sqrt{\omega^2/c^2 - \mathbf{k}_\perp^2}\right), \tag{3.179}$$

where the subscript $\perp$ denotes the components of the vector in the transverse x–y plane. Substituting this expression into Eq. (3.178) we find that

$$\begin{aligned} A(t; z, \mathbf{x}_\perp) &\equiv a(z, \mathbf{x}_\perp)e^{-i\omega t} \\ &= e^{-i\omega t}\int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} f(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\exp\left(\frac{iz}{c}\sqrt{\omega^2 - c^2k_\perp^2}\right). \end{aligned} \tag{3.180}$$

Because the time variation of a monochromatic wave is always $\exp(-i\omega t)$, we ignore this factor and concentrate on the spatial dependence of the amplitude,